Benha University
Faculty of Engineering- Shoubra
Eng. Mathematics & Physics Department
Preparatory Year

Final Term Exam
Date: 20 – 12 – 2014
Course: Mathematics 1 – A
Duration: **3** hours

- Answer **All** questions
- The Exam Consists of One page (تخلفات)
- No. of questions: 4
- Total Mark: 100 Marks

[1] Find **y`** from the following: 18

(a) $y = x^{\frac{2}{3}} + 2^{x^2}$ (b) $y = \sinh x^3 . \tan 3x$ (c) $y = \sin x^2 + \ln \sin 2x$

(d) $y = \cos^{-1} x^3 + \tan^{-2} x$ (e) $y^4 = x^3 + x^y$ (f) $y = t \sec t, \ x = t \cosh t$

[2](a)Find the following limits:

(i) $\lim\limits_{x\to 4} \dfrac{\sqrt{x} - 2}{x^2 + 16}$ (ii) $\lim\limits_{x\to 0} \dfrac{\ln^4(1 + x)}{2^x - 3^x}$ (iii) $\lim\limits_{x\to \pi} \dfrac{\sin 3x}{\tan 5x}$ (iv) $\lim\limits_{x\to \infty} \dfrac{x^3 + 2x}{x + x^2}$ 12

(b)Write the Maclurin's series of the function: $f(x) = x^2 + e^{2x}$. 5

(c)Show that : $\sinh^{-1} x = \ln(x + \sqrt{1 + x^2})$. 5

(d)Determine the extrema of : $f(x) = x + e^{-x}$ $g(x) = x - 2\ln x$ 10

[3] Integrate the following: 30

(a) $\int \dfrac{x^3}{(4 + x^2)^{3/2}} dx$ (b) $\int \dfrac{(x+3)}{x^2 + 4x + 13} dx$ (c) $\int \dfrac{(\ln x)^3}{x} dx$

(d) $\int \cos^4 x \sin^3 x \, dx$ (e) $\int x^3 \sqrt{(x^2 + 1)} \, dx$ (f) $\int e^{2x} \cosh 3x \, dx$

[4](a) Find the area of the surface of revaluation generated by revolving about ***x***-axis 7
the cycloid $x = a(\theta - \sin\theta), \ y = (1 - \cos\theta), \ 0 \le \theta \le 2\pi$

(b) Find the area bounded by the curves: $y = x^2, \ x = 2, \ x = 5, \ y = 0$. 7

(c) Find the volume generated by revolving, about x-axis, the area bounded by: 6
$x = a\cos\theta, \ y = a\sin\theta$

Good Luck *Dr. Mohamed Eid* *Dr. Fathi Abdsallam*

Model Answer

Answer Question (1)

[1](a) $y` = \frac{2}{3}x^{-\frac{1}{3}} - 2^{x^2}.\ln 2\,.\,2x$

(b) $y` = \cosh x^3.\,3x^2\,.\tan 3x + \sinh x^3\,.\sec^2 2x.\,2$

(c) $y = \cos x^2\,.\,2x + \frac{2\cos 2x}{\sin 2x}$

(d) $y` = \frac{-3x^2}{\sqrt{1-x^6}} - 2(\tan x)^{-3}.\sec^2 x$

(e) $4y^3.\,y` = 3x^2 + e^{y\ln x}(y`\ln x + \frac{y}{x})$

(f) $y` = \frac{\sec t + t\sec t.\tan t}{\cosh t + t.\sinh t}$

Answer Question (2)

(a)(i) $\text{Lim}_{x\to 4}\ \frac{\sqrt{x}-2}{x^2+16} = \frac{2-2}{16+16} = 0$

(ii) $\text{Lim}_{x\to 0}\ \frac{[\ln(1+x)]^4}{2^x-3^x} = \frac{0}{0} = \text{Lim}_{x\to 0}\ \frac{\ln(1+x)}{(\left(\frac{2}{3}\right)^x-1)}\frac{[\ln(1+x)]^3}{3^x} = \frac{1}{\ln 3}.\frac{0}{1} = 0$

(iii) $\text{Lim}_{x\to\pi}\ \frac{\sin 3x}{\tan 5x} = \frac{0}{0} = \text{Lim}_{x\to\infty}\frac{3\cos 3x}{5\sec^2 5x} = \frac{3}{5}$

(iv) $\text{Lim}_{x\to\infty}\ \frac{x^2+2x}{x+x^2} = \frac{\infty}{\infty} = \text{Lim}_{x\to\infty}\ \frac{2x+2}{1+2x} = \frac{\infty}{\infty} = \text{Lim}_{x\to\infty}\ \frac{2}{2} = 1$

(b) Since $f(0) = 1,\ f`(0) = 2,\ f``(0) = 6$

Then $f(x) = 1 + 2x + 3x^2 + \cdots$

(c)If $y = \sinh^{-1}x$. Then $x = \sinh y = \frac{1}{2}(e^y - e^{-y})$. Then $2x = e^y - e^{-y}$.

Then $e^{2y} - 2xe^y - 1 = 0$. Then $e^y = \frac{1}{2}[2x \pm \sqrt{(2x)^2 - 4(1)(-1)}] = \frac{1}{2}[2x \pm \sqrt{4x^2+4}]$

Then $e^y = x \pm \sqrt{x^2+1}$. Then $y = \ln[x + \sqrt{x^2+1}]$

(d) From $f`(x) = 1 - e^{-x} = 0$, then x = 0 and $f``(x) = e^{-x},\ f``(0) = 1$

Then, the point x = 0 is minimum.

From $g`(x) = 1 - \frac{2}{x} = 0$, then x = 2 and $g``(x) = 2x^{-2},\ g``(2) = 1$

Then, the point x = 2 is minimum.

Dr. Mohamed Eid

Answer Question (3)

(a) $put \ \ x = 2\tan\theta \quad \therefore dx = 2\sec^2\theta \, d\theta,$

$$\sqrt{4+x^2} = \sqrt{4+4\tan^2\theta} = 2\sec\theta, \quad \sec\theta = \sqrt{\tan^2\theta + 1} = \frac{1}{2}\sqrt{x^2+4}$$

Substitute in the problem we have

$$\therefore \int \frac{x^3}{(4+x^2)^{3/2}} dx = \int \frac{8\tan^3\theta \sec^2\theta \, d\theta}{8\sec^2\theta} = \int \frac{\tan^3\theta \, d\theta}{\sec\theta} = \int \frac{\tan^2\theta(\sec\theta\tan\theta)d\theta}{\sec^2\theta}$$

$$= \int \frac{(\sec^2\theta - 1)(\sec\theta\tan\theta)d\theta}{\sec^2\theta} = \int (1 - \frac{1}{\sec^2\theta})(\sec\theta\tan\theta)d\theta$$

$$= \int (\sec\theta\tan\theta - \frac{\sec\theta\tan\theta}{\sec^2\theta}) = \sec\theta + \frac{1}{\sec\theta} + c = \frac{1}{2}\sqrt{4+x^2} + \frac{2}{\sqrt{4+x^2}} + C$$

(b) $$\int \frac{(x+3)}{x^2+4x+13} dx = \frac{1}{2} \int \frac{(2x+6)}{x^2+4x+13} dx = \frac{1}{2} \int \frac{(2x+4+2)}{x^2+4x+13} dx$$

$$= \frac{1}{2} \int \frac{2x+4}{x^2+4x+13} dx + \frac{1}{2} \int \frac{2}{x^2+4x+13} dx$$

$$= \frac{1}{2} \int \frac{2x+4}{x^2+4x+13} dx + \int \frac{dx}{(x+2)^2+9} = \frac{1}{2} \ln\left|x^2+4x+13\right| + \frac{1}{3}\tan^{-1}\frac{(x+2)}{3}$$

(c) $\int \frac{(\ln x)^3}{x} dx \qquad$ Put $u = \ln x \quad \therefore du = \frac{1}{x} dx$

$$\int \frac{(\ln x)^3}{x} dx = \int u^3 du = \frac{1}{4}u^4 + c = \frac{1}{4}(\ln u)^4 + c$$

$$(d) \int \cos^4 x \sin^3 x \, dx = \int \cos^4 x \sin^2 x (\sin x dx) = \int \cos^4 x (1 - \cos^2 x)(\sin x dx)$$

$$= -\int (\cos^4 x - \cos^6 x)(-\sin x dx) = -\frac{1}{5}\cos^5 x - \frac{1}{7}\cos^7 x + c$$

(e) $$\int x^3\sqrt{x^2+1} \, dx = \int (x^3 + x - x)\sqrt{x^2+1} \, dx$$

$$= \int (x^3 + x)\sqrt{x^2+1} \, dx - \int x\sqrt{x^2+1} \, dx = \int x(x^2+1)\sqrt{x^2+1} \, dx - \int x\sqrt{x^2+1} \, dx$$

$$= \int x(x^2+1)^{\frac{3}{2}} dx - \int x\sqrt{x^2+1} \, dx = \frac{2}{10}(x^2+1)^{\frac{5}{2}} - \frac{2}{6}(x^2+1)^{\frac{3}{2}} + c$$

(f) $\int e^{2x}\cosh 3x\,dx=\int e^{2x}\dfrac{e^{3x}+e^{-3x}}{2}dx=\dfrac{1}{2}\int\left(e^{5x}+e^{-x}\right)dx=\left(\dfrac{1}{10}e^{5x}-\dfrac{1}{2}e^{-x}\right)+c$

Answer Question (4)

(a) $x=a(\theta-\sin\theta),\ y=a(1-\cos\theta),\ 0\le\theta\le 2\pi$

$$x=a(\theta-\sin\theta),\quad y=a(1-\cos\theta)\qquad \frac{dx}{d\theta}=a(1-\cos\theta),\qquad \frac{dy}{dt}=a\sin\theta$$

$$dL=\sqrt{\left[\frac{dx}{d\theta}\right]^2+\left[\frac{dy}{d\theta}\right]^2}\,dt=\sqrt{a^2(1-\cos\theta)^2+a^2\sin^2\theta}=2a\sin\frac{\theta}{2}d\theta$$

$$\therefore S_x=2\pi\int_a^b y\sqrt{\left[\frac{dx}{d\theta}\right]^2+\left[\frac{dy}{d\theta}\right]^2}\,d\theta=4\pi a\int_0^{2\pi}a(1-\cos\theta)\sin\frac{\theta}{2}d\theta$$

$$=8\pi a^2\int_0^{2\pi}\sin^3\frac{\theta}{2}d\theta=8\pi a^2\int_0^{2\pi}\sin^2\frac{\theta}{2}\sin\frac{\theta}{2}d\theta=8\pi a^2\int_0^{2\pi}(1-\cos^2\frac{\theta}{2})\sin\frac{\theta}{2}d\theta$$

$$=8\pi a^2\int_0^{2\pi}(\sin\frac{\theta}{2}-\cos^2\frac{\theta}{2}\sin\frac{\theta}{2})d\theta=-8\pi a^2\left[2\cos\frac{\theta}{2}+\frac{2}{3}\cos^3\frac{\theta}{2}\right]_0^{2\pi}=$$

$$=-8\pi a^2\left[(2\cos\frac{\pi}{2}+\frac{2}{3}\cos^3\frac{\pi}{2})-(2\cos 0+\frac{2}{3}\cos^3 0)\right]=-8\pi a^2\left[0-(2+\frac{2}{3})\right]=\frac{64}{3}a^2\pi$$

(b) $A=\int_2^5 y\,dx=\int_2^5 x^2dx=\left[\dfrac{1}{3}x^3\right]_2^5=\dfrac{1}{3}\left[5^3-2^3\right]=\dfrac{125-8}{3}=\dfrac{117}{3}=39$ *square unit*

(c) $V=\pi\int_{t=0}^{t=2\pi}y^2dx=\pi\int_{t=0}^{t=2\pi}(1-\cos t)^2(1-\cos t)dt=\pi\int_{t=0}^{t=2\pi}(1-\cos t)^3dt$

$$=\pi\int_{t=0}^{t=2\pi}(1-3\cos t+3\cos^2 t-\cos^3 t)dt=$$

$$=\pi\int_{t=0}^{t=2\pi}\left[\frac{5}{2}-3\cos t+\frac{3}{2}\cos 2t-(1-\sin^2 t)\cos t\right]dt$$

$$=\pi\left[\frac{5}{2}t-3\sin t+\frac{3}{4}\sin 2t-(\sin t-\frac{1}{3}\sin^3 t)\right]_0^{2\pi}=5\pi^2$$
