Benha University Faculty of Engineering (Shoubra) Engineering Mathematics and Physics Department		Ordinary and partial differential equations Code: EMM 401 28-12-2015 Time allowed: 3 hours Scores :200 marks

Question 1

- **Discuss the solution U(x,y) of the following P.D.E. analytically:**

i - $U_{tt} = c^2 U_{xx}$, $0 < x < L$, **with B.C. :** $U(0,t) = U(L,t) = 0$ **and I.C. :** $U(x,0) = f(x)$, $U_t(x,0) = g(x)$

ii - $U_t = c\, U_{xx}$, $0 < x < L$, **with B.C.:** $U(0,t) = U(L,t) = 0$ **and I.C. :** $U(x,0) = f(x)$.

- **Derive the suitable formulas for solving the above differential equations using finite difference**
- **If** $L = 1$, $f(x) = \sin \pi x$, $g(x) = x^2$, **solve the above differential equations numerically & analytically.**

Question 2

- **Find general solution of the following P.D.E. analytically**

1) $3u_x + 4\, u_y - 5u = 10\, y$

2) $4u_{xx} - 24\, u_{xy} + 11u_{yy} - 12\, u_x - 9u_y - 5\, u = 0$

- **Derive the general formula to obtain u(x,y) numerically using finite difference**

Question 3

Solve the I.V.P. using 2 different numerical methods

$$x` - 3y` = -2t + x - 2y - 7,\ 2x` + y` = 10t + y + 3 - t^2,\ x(0) = 1,\ y(0) = -3$$

Board of examiners Dr. eng Khaled El Naggar

Model answer

Answer of question 1

i) We use Separation method to solve the Wave equation, so that the solution is expressed as U(x,t) = $\phi(x)\Psi(t)$, therefore U_{xx}= $\phi''(x)\,\Psi(t)$ and U_{tt} = $\phi(x)\,\Psi''(t)$, thus $c^2\phi''(x)\,\Psi(t) = \phi(x)\,\Psi''(t)$.

Therefore $\frac{\phi''(x)}{\phi(x)} = \frac{1}{c^2}\frac{\psi''(x)}{\psi(x)} = -\lambda$, where λ is positive constant.

Thus $\phi''(x) + \lambda\,\phi(x) = 0$, the characteristic equation is $m^2 + \lambda = 0$, so

$\phi(x) = c_1 \cos\sqrt{\lambda}\,x + c_2 \sin\sqrt{\lambda}\,x$.

And $\Psi''(t) + c^2\lambda\,\Psi(t) = 0$, the characteristic equation is $n^2 + c^2\lambda = 0$, so

$\Psi(t) = c_3 \cos c\sqrt{\lambda}\,t + c_4 \sin c\sqrt{\lambda}\,t$.

Therefore $U(x,t) = (c_1 \cos\sqrt{\lambda}\,x + c_2 \sin\sqrt{\lambda}\,x)(c_3 \cos c\sqrt{\lambda}\,t + c_4 \sin c\sqrt{\lambda}\,t)$.

But U(0,t) = 0, therefore $c_1\,(c_3 \cos c\sqrt{\lambda}\,t + c_4 \sin c\sqrt{\lambda}\,t) = 0$, thus $c_1 = 0$, hence $U(x,t) = (c_2 \sin\sqrt{\lambda}\,x)(c_3 \cos c\sqrt{\lambda}\,t + c_4 \sin c\sqrt{\lambda}\,t)$.

Since U(L,t) = 0, therefore $(c_2 \sin\sqrt{\lambda}\,L)(c_3 \cos c\sqrt{\lambda}\,t + c_4 \sin c\sqrt{\lambda}\,t) = 0$, but $c_2 \neq 0$, thus $\sin\sqrt{\lambda}\,L$ = 0, hence $\sqrt{\lambda}\,L = n\pi \Rightarrow \lambda = (\frac{n\pi}{L})^2$, n = 1, 2, 3, …. Therefore $\phi(x) = (c_2 \sin(\frac{n\pi}{L})x)$, thus U(x,t) =

$$\sum_{n=1}^{\infty} \sin(\frac{n\pi}{L})x[A_n \cos(\frac{cn\pi}{L})t + B_n \sin(\frac{cn\pi}{L})t]$$

But U(x,0) = f(x) = $\sum_{n=1}^{\infty} A_n \sin(\frac{n\pi}{L})x$, which is Fourier sine series such that

$$A_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi}{L})x\,dx$$

Since $U_t(x,t) = \sum_{n=1}^{\infty}(\frac{cn\pi}{L})\sin(\frac{n\pi}{L})x[-A_n\sin(\frac{cn\pi}{L})t + B_n\cos(\frac{cn\pi}{L})t]$

And $U_t(x,0) = g(x)$, therefore $\sum_{n=1}^{\infty} B_n(\frac{cn\pi}{L})\sin(\frac{n\pi}{L})x = g(x)$, which is Fourier sine series such that $B_n(\frac{cn\pi}{L}) = \frac{2}{L}\int_0^L g(x)\sin(\frac{n\pi}{L})x\,dx$, therefore

$$B_n = \frac{2}{cn\pi}\int_0^L g(x)\sin(\frac{n\pi}{L})x\,dx$$

ii) We use Separation method to solve the Heat equation, so that the solution is expressed as $U(x,t) = \phi(x)\Psi(t)$, therefore $U_{xx} = \phi''(x)\,\Psi(t)$ and $U_t = \phi(x)\,\Psi'(t)$, thus $c\,\phi''(x)\,\Psi(t) = \phi(x)\,\Psi'(t)$.

Therefore $\frac{\phi''(x)}{\phi(x)} = \frac{1}{c}\frac{\psi'(x)}{\psi(x)} = -\lambda$, where λ is positive constant.

Thus $\phi''(x) + \lambda\,\phi(x) = 0$, the characteristic equation is $m^2 + \lambda = 0$, so

$\phi(x) = c_1\cos\sqrt{\lambda}\,x + c_2\sin\sqrt{\lambda}\,x$ and $\Psi'(t) + c\lambda\,\Psi(t) = 0$, the characteristic equation is $n + c\lambda = 0$, so $\Psi(t) = c_3\,e^{-c\lambda t}$

Therefore $U(x,t) = (c_1\cos\sqrt{\lambda}\,x + c_2\sin\sqrt{\lambda}\,x)(c_3\,e^{-c\lambda t})$.

But $U(0,t) = 0$, therefore $U(0,t) = (c_1)(c_3\,e^{-c\lambda t}) = 0$, thus $c_1=0$, hence $U(x,t) = (c_2\sin\sqrt{\lambda}\,x)(c_3\,e^{-c\lambda t})$, and $U(L,t) = 0$, therefore: $(c_2\sin\sqrt{\lambda}\,L)(c_3\,e^{-c\lambda t}) = 0$ and $c_2 \neq 0$, thus $\sin\sqrt{\lambda}\,L = 0$, hence $\sqrt{\lambda}\,L = n\pi \Rightarrow \lambda = (\frac{n\pi}{L})^2$, n = 1, 2, 3, …. Therefore $\phi(x) = (c_2\sin(\frac{n\pi}{L})x)$, thus $U(x,t) =$

$\sum_{n=1}^{\infty} A_n e^{(-\frac{cn\pi}{L})t} \sin(\frac{n\pi}{L})x$, but U(x,0) = f(x) = $\sum_{n=1}^{\infty} A_n \sin(\frac{n\pi}{L})x$, which is Fourier sine series such that $A_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi}{L})x\,dx$

- To solve the above equation numerically , we use graphical representation of partial equations such that: $u_x = \frac{u_{i+1,j} - u_{i,j}}{h} = \frac{u_{i,j} - u_{i-1,j}}{h}$, $u_y = \frac{u_{i,j+1} - u_{i,j}}{k} = \frac{u_{i,j} - u_{i,j-1}}{k}$,

$u_{xx} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$, $u_{yy} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2}$, $u_{xy} = \frac{u_{i+1,j+1} - u_{i-1,j+1} - u_{i+1,j-1} + u_{i-1,j-1}}{4hk}$, but U_{tt} = c^2 U_{xx} , therefore $\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = c^2 \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$ and for U_t = c U_{xx} , therefore $\frac{u_{i,j+1} - u_{i,j}}{k} = c\,\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$

Answer of question 2

1) Consider homogeneous partial differential equation:

$$au_x + b\,u_y + cu = g(x,y)$$

The solution will be in the form: $w_\alpha + k\,w = g(\alpha, \beta)$ such that:

$\alpha = x\cos\theta + y\sin\theta$, $\beta = -x\sin\theta + y\cos\theta$

$u_x = u_\alpha\ \alpha_x + u_\beta\ \beta_x = u_\alpha \cos\theta - u_\beta \sin\theta$,

$u_y = u_\alpha\ \alpha_y + u_\beta\ \beta_y = u_\alpha \sin\theta + u_\beta \cos\theta$

Substitute in the above P.D.E. so that:

a $(w_\alpha \cos\theta - w_\beta \sin\theta) + b(w_\alpha \sin\theta + w_\beta \cos\theta) + cw = w(\alpha,\beta)$

Coefficient of $w_\beta = 0$, therefore $-a\sin\theta + b\cos\theta = 0$, thus $\tan\theta = b/a$, $\sin\theta = \frac{b}{\sqrt{a^2+b^2}}$ and $\cos\theta = \frac{a}{\sqrt{a^2+b^2}}$

Therefore $\sqrt{a^2+b^2}\; w_\alpha + cw = w(\alpha,\beta)$ which is linear D.E. such that the solution will be in the form:

$$we^{\int \frac{c}{\sqrt{a^2+b^2}} d\alpha} = \int \frac{w(\alpha,\beta)}{\sqrt{a^2+b^2}} e^{\int \frac{c}{\sqrt{a^2+b^2}} d\alpha} d\alpha + \phi(\beta)$$

Where a = 3, b = 4, c = -5 and g(x,y) = 10y, $\sin\theta = \frac{4}{5}$, $\cos\theta = \frac{3}{5}$, and $5\alpha = 3x + 4y$, $5\beta = -4x + 3y$ therefore $w(\alpha,\beta) = 10y = 8\alpha + 6\beta \Rightarrow 5w_\alpha - 5w = 8\alpha + 6\beta$ $\Rightarrow w\,e^{-\alpha} = \int \frac{8\alpha + 6\beta}{5} e^{-\alpha} d\alpha + \phi(\beta)$.

2) $au_{xx} + b\,u_{xy} + cu_{yy} + hu_x + k\,u_y + eu = 0$

Let the general solution of the above equation is $u(x,y) = f(x+\lambda y) = f(v)$, where $v = x + \lambda y$, thus

$$u_x = \frac{du}{dv} v_x = \frac{du}{dv},\; u_{xx} = \frac{d^2u}{dv^2}(v_x)^2 + \frac{du}{dv} v_{xx} = \frac{d^2u}{dv^2},$$

$$u_{xy} = \frac{d^2u}{dv^2} v_x v_y + \frac{du}{dv} v_{xy} = \frac{d^2u}{dv^2}\lambda$$

$$u_y = \frac{du}{dv} v_y = \frac{du}{dv}\lambda,\; u_{yy} = \frac{d^2u}{dv^2}(v_y)^2 + \frac{du}{dv} v_{yy} = \frac{d^2u}{dv^2}\lambda^2$$

By taking the homogeneous part $au_{xx} + b\,u_{xy} + cu_{yy} = 0$, we will get

$a + b\lambda + c\lambda^2 = 0$, where λ_1, λ_2 are the two roots of the characteristic equation.

Where a = 4, b = -24, c = 11, h = -12, k = -9, e = -5, therefore $11\lambda^2 - 24\lambda + 4=0$, hence $\lambda = 2$ or $\lambda = 2/11$ from which the general solution of the above equation is $u(x,y) = c_1 f(x+ 2y) + c_2 f(x+ (2/11)y)$

- To solve the above equation numerically , we use graphical representation of partial equations such that: $u_x = \frac{u_{i+1,j} - u_{i,j}}{h} = \frac{u_{i,j} - u_{i-1,j}}{h}$, $u_y = \frac{u_{i,j+1} - u_{i,j}}{k} = \frac{u_{i,j} - u_{i,j-1}}{k}$,

$u_{xx} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$, $u_{yy} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2}$, $u_{xy} = \frac{u_{i+1,j+1} - u_{i-1,j+1} - u_{i+1,j-1} + u_{i-1,j-1}}{4hk}$, but $au_x + b\,u_y + cu = g(x,y)$, therefore $a\left[\frac{u_{i+1,j} - u_{i,j}}{h}\right] + b\left[\frac{u_{i,j+1} - u_{i,j}}{k}\right] + c\,u_{i,j} = g(x_i,y_j)$ and for $a\,u_{xx} + b\,u_{xy} + c\,u_{yy} + h\,u_x + k\,u_y + e\,u = 0$, therefore $a\left[\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}\right] + b\left[\frac{u_{i+1,j+1} - u_{i-1,j+1} - u_{i+1,j-1} + u_{i-1,j-1}}{4hk}\right] + c\left[\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2}\right] + h\left[\frac{u_{i+1,j} - u_{i,j}}{h}\right] + k\left[\frac{u_{i,j+1} - u_{i,j}}{k}\right] + e\,u_{i,j} = 0$

Answer of question 3

$x` = x^2 – 2tx + y – 2t = f(x,y,t)$, $y` = y-x^2 – 2tx + 2t + 3 = \varphi(x,y,t)$, $x_0 = 2$, $y_0 = 3$, $t_0 = 1$

$y_{i+1} = y_i + (h/2)[\varphi(t_i, x_i, y_i) + \varphi(t_{i+1}, x_i + hf(t_i, x_i, y_i), y_i + h\varphi(t_i, x_i, y_i))]$

Put i = 0, therefore $y_1 = y_0 + (h/2)[\varphi(t_0, x_0, y_0) + \varphi(t_1, x_0 + hf(t_0, x_0, y_0), y_0 + h\varphi(t_0, x_0, y_0))] = 2.9585$

2-ii) $y_{n+1} = y_0 + \int_{t_0}^{t} (x_n z_n + 28x_n - y_n)\,dt$, $x_{n+1} = x_0 + \int_{t_0}^{t} -10(x_n - y_n)\,dt$, $z_{n+1} = z_0 + \int_{t_0}^{t} (x_n y_n - 8z_n/3)\,dt$, $y_0 = -1$, $x_0 = 2$, $t_0 = 0$, $z_0 = 3$, thus $x_1 = x_0 + \int_{t_0}^{t} -10(x_0 - y_0)\,dt$, $y_1 = y_0$

$+\int_{t_0}^{t}(x_0z_0 + 28x_0 - y_0)\,dt$ and $z_1 = z_0 + \int_{t_0}^{t}(x_0y_0 - 8z_0/3)\,dt$, therefore $x_1 = 2-30t$, $y_1=-1+51t$, $z_1 = 3-10t$. Similarly, $x_2=x_0+\int_{t_0}^{t} -10(x_1 - y_1)\,dt$, $y_2 = y_0 + \int_{t_0}^{t}(x_1z_1 + 28x_1 - y_1)\,dt$ and $z_2= z_0 + \int_{t_0}^{t}(x_1y_1 - 8z_1/3)\,dt$, therefore $x_2 = 2 - 30t + 405t^2$, $y_2 =-1+51t-(781/2)t^2 - 100t^3$, $z_2 = 3-10t +(238/3)t^2 - 510t^3$.

2nd : <u>using Euler</u>, $x_{n+1} = x_n + h\,[-10(x_n - y_n)]$, $y_{n+1} = y_n + h\,[-x_n z_n + 28\,x_n - y_n]$, thus $x_1 =x_0 +h[-10(x_0 - y_0)] = 0.5= x(0.05)$, $y_1 = y_0 +h\,[-x_0\,z_0 + 28\,x_0 - y_0] =1.55 = y(0.05)$, therefore $x(0.1) = x_2 = x_1 +h[-10(x_1 - y_1)] = 1.025$

1. **Overall aims of course**

By the end of the course the students will be able to:

- Solve ordinary and partial differential equations numerically
- Recognize finite difference method in solving P.D.E.
- Describe error analysis and stability for P.D.E.

2. **Intended Learning outcomes of Course (ILOs)**

a. Knowledge and Understanding:

2.1.1 Identify theories, fundamentals of ordinary and partial differential equations [**Q1, Q2, Q3**]

2.1.3 Recognize the developments of finite difference method in solving P.D.E. [**Q1, Q2**]

2.1.4 Summarize the moral and legal principles of error analysis and stability [**Q1, Q2**]

b. Intellectual Skills

2.2.5 Assess solutions of partial differential equations using finite difference method. [**Q1, Q2**]

c. Professional and Practical Skills

2.3.1 Interpret professional skills in estimating error analysis and stability. [**Q1, Q2**]

d. General and Transferable Skills

2.4.1 Communicate effectively using researches of new topics about solutions of ordinary and partial differential equations .

2.4.5 Assess the performance of error analysis and stability

2.4.6 Work in a group and manage time effectively